Mathematical Modelling, Vol. 1, pp. 255–270, 1980
0270-0255/80/030255-16$02.00/0

CONTINUATION OF SOLUTIONS OF CONSTRAINED EXTREMUM PROBLEMS AND NONLINEAR EIGENVALUE PROBLEMS

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Communicated by Gian-Carlo Rota

Abstract—In this paper we continue our investigations, begun in the previous paper, of describing the solution sets of constrained extremum problems

$$\inf_{u \in t^{-1}(p)} f(u), \tag{*}$$

where f and t are twice continuously differentiable functionals on a reflexive Banach space V, and $t^{-1}(p)$ denotes the level set of the functional t with value $p \in \mathbf{R}$.

Considering p as a parameter in (*) we obtain results concerning the continuation of solutions of (*) and consequently also concerning specific solution branches of the nonlinear eigenvalue problem

$$f'(u) = \mu t'(u). \tag{**}$$

The general results are applied to functionals which lead to nonlinear eigenvalue problems of a semilinear elliptic type and in particular we consider a specific example for which there occurs "bending" of a solution curve (u,μ) of (**).

1. PRELIMINARIES AND NOTATION

Let V be a reflexive Banach space, V^* its dual and $\langle\ ,\ \rangle$ the duality map. We consider two functionals f and t defined on V which we assume to satisfy

($f1$) f is weakly lower semicontinuous,
and f is coercive on V [i.e., $f(u) \to \infty$ if $\|u\|_V \to \infty$].
($t1$) t is weakly continuous.
$c(f,t2)$ $f,t \in C^2(V,\mathbf{R})$.

We consider *level sets* of the functional t:

$$t^{-1}(p) := \{u \in V \mid t(u) = p\}$$

for $p \in t(V)$ where $t(V)$ denotes the range of the functional t. The *tangent space* at a point $\hat{u} \in t^{-1}(t(\hat{u}))$ is defined if $t'(\hat{u}) \neq 0$ as

$$\tau_{\hat{u}} := \{v \in V \mid \langle t'(\hat{u}), v\rangle = 0\} \subset V.$$

Writing

$$n^* := i'(u) \in V^*,$$

any element $n \in V$ with $\langle n,n^*\rangle = 1$ can be taken as a normal to the tangent plane $u + \tau_u$. Having chosen a normal n, the dual tangent space is defined as

$$\tau_u^* := \{v^* \in V^* \mid \langle n,v^*\rangle = 0\}.$$

Then we have the topological direct sum representation:

$$V = \tau_u + \{n\},\ V^* = \tau_u^* + \{n^*\}.$$

Correspondingly, we can define projection operators:

$$P_u : V \to \tau_u \quad : P_u\phi := \phi - \langle \phi,n^*\rangle n \qquad \text{for } \phi \in V$$
$$P_u^* : V^* \to \tau_u^* : P_u^*\phi^* = \phi^* - \langle \phi^*,n\rangle n^* \quad \text{for } \phi^* \in V^*.$$

The first two lemmas will be useful in the following when we consider the linearized operator and the second variation on the tangent space.

Lemma 1.1. Suppose $V = \tau + \{n\}$, and let $Q : V \to V^*$ be a linear and self-adjoint operator. Then we have, if Q is positive definite on τ, i.e., if for some constant $c > 0$

$$\langle Qv,v\rangle \geq c \| v \| \qquad \forall v \in \tau,$$

then the operator

$$P^*Q : \tau \to \tau^*$$

is boundedly invertible.

Proof. We have to show that for every $\psi \in V^*$ there exists a unique $v \in \tau$ such that $P^*(Qv - \psi) = 0$. Therefore consider

$$\inf_{\substack{v \in V \\ \langle v,n^*\rangle = 0}} \{\tfrac{1}{2}\langle Qv,v\rangle - \langle \psi,v\rangle\}.$$

By a standard result, this minimization problem has a unique solution v which satisfies for some $\gamma \in \mathbf{R}$ the equation

$$Qv - \psi = \gamma n^* .$$

Applying the projection operator P^* to this equation and using $P^*n^* = 0$ it follows that $P^*(Qv - \psi) = 0$.

Lemma 1.2. Let $V = \tau + \{n\}$ and let $q : V \to \mathbf{R}$ be a quadratic functional on V. Suppose

(i) q is weakly lower semicontinuous on V,

(ii) q is positive on τ: $q(v) > 0$ for $v \in \tau \setminus \{0\}$.

(iii) q satisfies the following condition c.c.: for every sequence $v_n \in V$ for which $q(v_n) \to 0$ and for which $v_n \rightharpoonup 0$ (weakly) in V, it follows that $v_n \to 0$ (strongly) in V.

Then the functional q is positive definite on τ, i.e., there exists a constant $c > 0$ such that

$$q(v) \geq c \cdot \| v \|^2 \qquad \forall v \in \tau.$$

Proof. Suppose the conclusion does not hold. Then there exists a sequence $\{v_n\} \subset V$ such that $\| v_n \| = 1$, $v_n \in \tau$, $q(v_n) \to 0$ as $n \to \infty$. As $\{v_n\}$ is uniformly bounded in V, there exists a weakly convergent subsequence, say $v_n \rightharpoonup \hat{v}$ and moreover $\hat{v} \in \tau$. Then, by (i), $q(\hat{v}) \leq \liminf q(v_n) = 0$, and then because of (ii), $\hat{v} = 0$. Hence $q(v_n) \to 0$ and $v_n \rightharpoonup 0$. Then by condition c.c. $v_n \to 0$, which contradicts the assumption $\| v_n \| = 1$.

We now come to the study of the operator equation in V^*:

$$f'(u) = \mu t'(u) \qquad u \in V, \mu \in \mathbf{R}, \tag{1.1}$$

where $f', t' : V \to V^*$ denote the derivatives of the functionals f and t, respectively. In particular, we shall study (1.1) in relation with solutions of the constrained extremum problems:

$$\mathcal{P}_p : \inf_{u \in t^{-1}(p)} f(u). \tag{1.2}$$

A solution of the nonlinear eigenvalue problem (1.1) will be a couple $(u,\mu) \in V \times \mathbf{R}$ which satisfies (1.1). A solution of $\mathcal{P}_p$ will be any element $\hat{u} \in V$ for which

$$t(\hat{u}) = p \quad \text{and} \quad f(\hat{u}) \leq f(u) \qquad \forall u \in t^{-1}(p).$$

The next lemma resumes some well known results. (See, e.g., Vainberg [7, Theorem 9.11], Berger [1, Sec. 3.1F]).

Lemma 1.3.

(a) For every $p \in t(V)$, problem $\mathcal{P}_p$ has at least one solution.

(b) If u is a solution of $\mathcal{P}_p$ for which $t'(u) \neq 0$, then there exists a unique multiplier $\mu \in \mathbf{R}$ such that (u,μ) is a solution of (1.1).

(c) In the situation of (b), the second variation is nonnegative on the tangent space τ_u:

$$\langle (f''(u) - \mu t''(u))v,v \rangle \geq 0 \qquad \forall v \in \tau_u.$$

Notation: for given $u \in V$, $\mu \in \mathbf{R}$ we shall write

$$Q(u,\mu) := f''(u) - \mu t''(u). \tag{1.3}$$

Then $Q(u,\mu) : V \to V^*$ is a self-adjoint operator.

In the following, the first and second eigenvalue of $Q(u,\mu)$, and a number ν will play an important role. Therefore let us suppose that

$$V \subset H \subset V^*,$$

where H is a Hilbert space for which the duality map $\langle \ , \ \rangle$ is the inner product. In that case, if $Q : V \to V^*$ is a self-adjoint mapping, and if σ_1 denotes the principal (smallest say) eigenvalue of Q (with eigenfunction ϕ_1) and σ_2 the eigenvalue following σ_1, then we have

$$\sigma_1 = \min_{\phi \in V} \frac{\langle Q\phi,\phi\rangle}{\langle \phi,\phi\rangle} = \frac{\langle Q\phi_1,\phi_1\rangle}{\langle \phi_1,\phi_1\rangle} \tag{1.4}$$

$$\sigma_2 = \min_{\substack{\phi \in V \\ \langle \phi,\phi_1\rangle = 0}} \frac{\langle Q\phi,\phi\rangle}{\langle \phi,\phi\rangle} = \max_{w \in V} \inf_{\substack{\phi \in V \\ \langle \phi,w\rangle = 0}} \frac{\langle Q\phi,\phi\rangle}{\langle \phi,\phi\rangle}. \tag{1.5}$$

If $V = \tau + \{n\}$ we define a number ν by

$$\nu := \inf_{\phi \in \tau} \frac{\langle Q\phi,\phi\rangle}{\langle \phi,\phi\rangle}. \tag{1.6}$$

Clearly from the extremal characterizations

$$\sigma_1 \leq \nu \leq \sigma_2. \tag{1.7}$$

If $Q = Q(u,\mu)$, the dependence of these numbers σ_1, σ_2 and ν will be expressed by writing $\sigma_1(u,\mu)$, $\sigma_2(u,\mu)$, and $\nu(u,\mu)$. The contents of Lemma 1.3(c) can then be rephrased: If $u(p)$ is a solution of problem $\mathscr{P}_p$ with $\mu(p)$ as multiplier, then we have

$$\nu[u(p),\mu(p)] \geq 0.$$

With Lemma's 1.1 and 1.2 we arrive at the following conclusion:

Lemma 1.4. Let u be a solution of problem $\mathscr{P}_p$, with μ as multiplier. Suppose that the quadratic functional

$$V \ni v \to \langle Q(u,\mu)v,v\rangle$$

satisfies the compactness condition c.c. of Lemma 1.2. Suppose, moreover, that instead of $\nu(u,\mu) \geq 0$ we know that

$$\nu(u,\mu) > 0.$$

Then the operator $Q(u,\mu)$ restricted to the tangent space τ_u is boundedly invertible.

2. LOCAL CONTINUATION

In this section we consider the continuation of solutions of the nonlinear eigenvalue problem

$$f'(u) = \mu t'(u). \tag{2.1}$$

Usually, if $(u_0,\mu_0) \in V \times \mathbf{R}$ is a solution of (2.1) one looks for a continuation parameterized with the number $\mu \in \mathbf{R}$. Applying the implicit function theorem to the equation $\Phi(u,\mu) = 0$, where Φ is the mapping

$$\Phi : V \times \mathbf{R} \to V^* \ : \Phi(u,\mu) = f'(u) - \mu t'(u),$$

it is a standard result that if $Q(u,\mu) : V \to V^*$ is boundedly invertible on V, then such a continuation is possible: i.e., there exists a number $\delta > 0$ such that $\{u(\mu),\mu\}_{\mu\in(\mu_0-\delta,\mu_0+\delta)}$ defines a (unique) curve in $V \times \mathbf{R}$ through the point (u_0,μ_0), where $(u(\mu),\mu)$ is a solution of (2.1) and u depends continuously on $\mu \in (\mu_0 - \delta, \mu_0 + \delta)$. Our aim is to show that for nonlinear eigenvalue problems with potential operators such as (2.1), a continuation with another parameter [in fact the value $p = t(u)$] is in many cases more appropriate.

The following result was motivated for the extremal solutions of problem $\mathcal{P}_p$ (in which case Lemma 1.4 may be useful for the applicability of the following result), but it may also be useful for other solutions of (2.1) [not necessarily constrained extremal solutions of $\mathcal{P}_p$, but merely stationary points of f with respect to the level set $t^{-1}(p)$].

THEOREM 2.1. *Let $(u_0,\mu_0) \in V \times \mathbf{R}$ be a solution of (2.1) and put $p_0 := t(u_0)$. Suppose that*
(i) $t'(u_0) \neq 0$
(ii) The restriction of $Q(u_0,\mu_0) = f''(u_0) - \mu_0 t''(u_0)$ to the tangent space $\tau_{u_0} = \{v \in V \mid \langle t'(u_0),v\rangle = 0\}$ is boundedly invertible.
Then there exists a unique continuation of the solution (u_0,μ_0) of (2.1), which continuation is smooth and may be parameterized with the parameter $p = t(u)$. More precisely: there exists a number $\delta > 0$ and a unique, continuously differentiable mapping

$$(p_0 - \delta, p_0 + \delta) \ni p \to [u(p),\mu(p)] \in V \times \mathbf{R}$$

which satisfies

$$f'(u(p)) = \mu(p) \cdot t'(u(p)), \quad t(u(p)) = p$$

and

$$u(p_0) = u_0, \ \mu(p_0) = \mu_0.$$

Proof. Consider the mapping

$$F : V \times \mathbf{R} \times \mathbf{R} \to V^* \times \mathbf{R} : F(u,\mu,p) := \begin{pmatrix} f'(u) - \mu t'(u) \\ t(u) - p \end{pmatrix}.$$

We shall use the implicit function theorem to show that the equation

$$F(u,\mu,p) = 0$$

has a solution curve $(u(p),\mu(p))$ as required. Therefore we need to verify the conditions of the implicit function theorem: (a) By assumption $F(u_0,\mu_0,p_0) = 0$, and (b) F is continuously differentiable with respect to its arguments. (c) Let $D_{u,\mu}F$ denote the derivative of F with respect to the variables (u,μ). Then, for $\xi \in V$ and $\alpha \in \mathbf{R}$:

$$D_{u,\mu}F(u,\mu,p)(\xi,\alpha) = \begin{pmatrix} Q_0\xi - \alpha t'(u) \\ \langle t'(u),\xi\rangle \end{pmatrix},$$

where $Q_0 := f''(u_0) - \mu_0 t''(u_0)$. In order to prove that $D_{u,\mu}F(u_0,\mu_0,p_0) : V \times \mathbf{R} \to V^*$

$\times$ **R** is boundedly invertible, we shall show that for arbitrary $u^* \in V^*$ and $\gamma \in$ **R** there exists a unique solution $(\xi,\alpha) \in V \times$ **R** of the equations

$$Q_0\xi - \alpha t'(u_0) = u^* \tag{$\dagger$}$$

$$\langle t'(u_0),\xi\rangle = \gamma. \tag{$\ddagger$}$$

Because of (i) it is possible to take a normal n to τ_{u_0} such that $\langle t'(u_0),n\rangle = 1$. Then, if we write $\xi = \beta n + v$ with $v \in \tau_{u_0}$, we have to find unique values α and β and a unique $v \in \tau_{u_0}$.

From Eq. ($\ddagger$) it follows that $\beta = \gamma$; then $\xi = \gamma n + v$ satisfies ($\ddagger$) for arbitrary $v \in \tau_{u_0}$. Projecting equation ($\dagger$) onto $\tau^*_{u_0} = \{v^* \in V^* \mid \langle n, v^*\rangle = 0\}$ gives rise to an equation for v:

$$P^* Q_0 v = P^* (u^* - Q_0\gamma n).$$

Because of (ii) this equation has a unique solution $v \in \tau_{u_0}$. Finally applying $I - P^*$ to Eq. ($\dagger$), i.e., taking the inner product of ($\dagger$) with the normal n, uniquely determines the value of α:

$$\begin{aligned}\alpha &= \langle Q_0(\gamma n+v),n\rangle - \langle u^* ,n\rangle \\ &= \gamma\langle Q_0 n,n\rangle - \langle u^* ,n\rangle.\end{aligned}$$

Having verified the necessary conditions for the applicability, the implicit function theorem (see, e.g., Dieudonné [3], Theorem 10.2.1) immediately leads to the results formulated by the theorem.

Lemma 2.2. In the situation of Theorem 2.1 we have the following additional information for $p \in (p_0 - \delta, p_0 + \delta)$:
Let

$$n^* (p) := t'(u(p)) \quad\text{and}\quad n(p) := \frac{\mathrm{d}u}{\mathrm{d}p}(p). \tag{2.2}$$

Then

$$\langle n(p),n^* (p)\rangle = 1, \tag{2.3}$$

i.e., $n(p)$ is normal to the tangent space $\tau_{u(p)}$, and we have

$$V = \tau_{u(p)} + \{n(p)\}, \quad V^* = \tau^*_{u(p)} + \{n^* (p)\}. \tag{2.4}$$

The mapping $Q_p := Q(u(p),\mu(p))$ satisfies ($P^* : V^* \to \tau^*_{u(p)}$ is the projection operator onto $\tau^*_{u(p)}$)

$$P^* Q_p : \tau_{u(p)} \to \tau^*_{u(p)} \text{ is boundedly invertible}, \tag{2.5}$$

$$Q_p : \{n(p)\} \to \{n^* (p)\}, \text{ in fact } Qn(p) = \frac{\mathrm{d}\mu}{\mathrm{d}p}(p)\cdot n^* (p), \tag{2.6}$$

and for the (point-) spectrum of Q_p we have

$$\sigma(Q_p) = \sigma(Q_p|_{\tau_{u(p)}}) + \omega(p) \tag{2.7}$$

where $Q_p|_{\tau_{u(p)}}$ denotes the restriction of Q_p to $\tau_{u(p)}$, and

$$\omega(p) := \frac{d\mu}{dp}(p) \cdot \langle n(p), n(p)\rangle^{-1}. \tag{2.8}$$

In particular, for the smallest eigenvalue $\sigma_1(p)$ of Q_p we have

$$\sigma_1(p) = \min[\nu(p), \omega(p)], \tag{2.9}$$

where $\nu(p)$ is defined by (1.6).

Proof. If we differentiate $t(u(p)) = p$ with respect to p there results

$$\langle t'(u(p)), \frac{du}{dp}(p)\rangle = 1, \tag{2.10}$$

which is (2.3), and by differentiating $f'(u(p)) = \mu(p)t'(u(p))$ with respect to p we obtain

$$Q_p \frac{du}{dp}(p) = \frac{d\mu}{dp}(p) \cdot t'(u(p)), \tag{2.11}$$

which is (2.6). To show (2.7) it suffices to note that, using (2.6),

$$\langle Qv, n\rangle = \langle v, Qn\rangle = \frac{d\mu}{dp}(p) \cdot \langle v, n^*\rangle = 0 \text{ for } v \in \tau_{u(p)}, \tag{2.12}$$

such that for $\alpha \in \mathbf{R}$ and $v \in \tau_{u(p)}$:

$$\langle Q_p(\alpha n + v), \alpha n + v\rangle = \alpha^2\langle Qn, n\rangle + \langle Qv, v\rangle. \tag{2.13}$$

Remark 2.3. "*p-parameterization*" versus "*μ-parameterization*."

From the foregoing results it will have become clear why in some cases a continuation described with the parameter p is more appropriate than with the parameter μ:

$$\text{if } 0 \in \sigma(Q_{p_0}|_{\tau_{u(p_0)}})$$

then both continuation methods fail to be applicable (at least it cannot be proved, with the usual implicit function theorem-methods, that such a continuation is possible);

$$\text{if } 0 \notin \sigma(Q_{p_0}|_{\tau_{u(p_0)}})$$

a continuation with the parameter p is possible no matter the value of $\omega(p_0)$, whereas if $\omega(p_0) = 0$ continuation with the parameter μ cannot be proved or is not possible because of the fact that

$$\frac{d\mu}{dp}(p_0) = 0,$$

together with a change of sign of $\frac{d\mu}{dp}$ at p_0.

Let us briefly describe a specific case of this last situation. Suppose

$$\sigma_1(p) > 0 \quad \text{for } p < p_0$$

$$\sigma_1(p) < 0 \quad \text{for } p > p_0$$

$$\nu(p_0) > 0$$

then clearly, $\omega(p)$ crosses zero at p_0 : $\omega(p_0) = 0$ which means that, if $t'(u_0) \neq 0$, $(\mathrm{d}\mu/\mathrm{d}p)(p_0) = 0$, $(\mathrm{d}\mu/\mathrm{d}p)(p) > 0$ for $p < p_0$ and $(\mathrm{d}\mu/\mathrm{d}p)(p) < 0$ for $p > p_0$. Hence $\mu(p)$ has a maximum at $p = p_0$, and the mapping $\mu \to u(\mu)$ will become multivalued in a left-neighbourhood of $\mu(p_0)$, whereas continuation with the parameter p causes no difficulties. In a usual bifurcation diagram (see Fig. 1) the situation is as indicated and is known as "bending of the solution curve." A specific example of this phenomenon will be considered in Sec. 5.

Remark 2.4. Let u_0 be a solution of $\mathcal{P}_{p_0}$ [i.e., u_0 is a global constrained minimum of f on $t^{-1}(p_0)$] with μ_0 as multiplier, and suppose that a continuation as described in Theorem 2.1 is possible. In particular, suppose $\nu(p_0) > 0$. Then, by continuity, $\nu(p) > 0$ in a sufficiently small neighbourhood of p from which it follows that the elements $u(p)$ of the curve $(u(p),\mu(p))$ through (u_0,μ_0) are *local* constrained minima of f on $t^{-1}(p)$ (for $|p - p_0|$ sufficiently small). However there is no evidence at all that these solutions $u(p)$ for $p \neq p_0$ are also *global* constrained minima of f on $t^{-1}(p)$. In other words, these functions $u(p)$ need not be solutions of the extremum problems $\mathcal{P}_p$. To investigate this and related problems we shall study some global aspects of problems $\mathcal{P}_p$ in the next section after which the results of the local and the global investigations can be glued together.

3. SOME GLOBAL ASPECTS OF CONSTRAINED EXTREMUM PROBLEMS

In this section we shall describe some results from van Groesen [5] and derive some more information concerning the solution sets of the constrained extremum problems $\mathcal{P}_p$ (recall the existence result and the multiplier rule for solutions of $\mathcal{P}_p$, Lemma 1.3). Define the function

$$h(p) := \inf_{u \in t^{-1}(p)} f(u) \qquad \text{for } p \in t(V). \tag{3.1}$$

In another paper [5] the following relation between this function and the multiplier of a solution of $\mathcal{P}_p$ was derived.

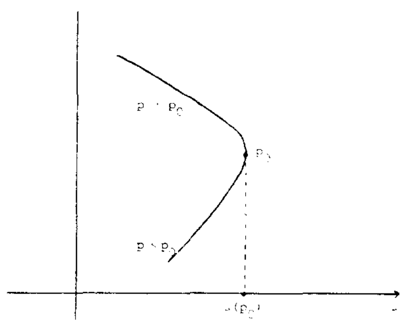

Fig. 1.

Lemma 3.1. Let u be any solution of $\mathscr{P}_p$ with $t'(u) \neq 0$. Then, if μ is the multiplier corresponding to u, we have

$$h'_+(p) \leq \mu \leq h'_-(p), \tag{3.2}$$

where h'_+ and h'_- denote the right- and left-hand-side derivative of h, respectively.

Corollaries 3.2.

(a) If h is differentiable at p, then

$$\frac{\mathrm{d}h}{\mathrm{d}p}(p) =: \mu(p) \tag{3.3}$$

is the unique multiplier for every solution of $\mathscr{P}_p$.

(b) Suppose that in some interval $J \subset t(V)$ h is known to be a convex function. Then h is differentiable at every $p \in J$ and hence (3.3) holds.

Remark 3.3. In another paper [5] it was shown that if h is convex in a neighbourhood of a point p, then $\mathscr{P}_p$ is equivalent to an *unconstrained* local extremum problem in the following sense: if $\hat{u}$ is any solution of $\mathscr{P}_p$ [and, necessarily, $\mu(p) = (\mathrm{d}h/\mathrm{d}p)(p)$ its multiplier] then there exists a neighbourhood $\Omega(\hat{u}) \subset V$ of $\hat{u}$ such that $\hat{u}$ is the unique solution of

$$\inf_{u \in \Omega(\hat{u})} \{f(u) - \mu(p)t(u)\}. \tag{3.4}$$

Moreover, if h is known to be subdifferentiable at p, then $\Omega(\hat{u})$ may be taken to be all of V: in this case $\hat{u}$ is a global minimum point of the functional $f - \mu(p)t$.

For what follows it is convenient to define the solution sets of problems $\mathscr{P}_p$:

$$P_p := \{u \in V \mid u \text{ is a solution of } \mathscr{P}_p\}. \tag{3.5}$$

The following theorem extends the results of Proposition 2.2 in Ref. 5.

THEOREM 3.4. *Let* $[\alpha,\beta]$ *be a closed and bounded interval of* $t(V)$ *and suppose that* h *is differentiable on* (α,β) *with* $h'_+(\alpha)$ *and* $h'_-(\beta)$ *finite. Then, in general* [*i.e., f and t satisfy* (*f,t,1*) *and* (*f,t,2*)], *we have*

$$\bigcup_{q \in [\alpha,\beta]} P_p \text{ is a weakly compact subset of } V. \tag{3.6}$$

Moreover, if f satisfies the extra condition

(*f3*) *for every sequence* u_n *for which* $u_n \rightharpoonup u$ (*weakly*) *in* V *and for which* $f(u_n) \to f(\hat{u})$ *it follows that* $u_n \to \hat{u}$ (*strongly*) *in* V,

then

$$\bigcup_{q \in [\alpha,\beta]} P_p \text{ is a compact subset of } V. \tag{3.7}$$

Proof. Let $\{u_n\}$ be any sequence in

$$\bigcup_{q \in [\alpha,\beta]} P_q,$$

and let $q_n := t(u_n)$. As $[\alpha,\beta]$ is a compact interval, for some subsequence, again to be denoted by u_n, we have $q_n \to p \in [\alpha,\beta]$. We have to show that there exists a subsequence $\{u_{n'}\}$ such that $u_{n'} \rightharpoonup \hat{u}$ [and, in case condition (f3) is satisfied, $u_n \to \hat{u}$] where

$$\hat{u} \in \bigcup_{q\in[\alpha,\beta]} P_q.$$

For the sequence $\{u_n\}$ we have $t(u_n) = q_n$ and $f(u_n) = h(q_n)$. As h is bounded on $[\alpha,\beta]$, $\{f(u_n)\}$ is bounded and thus, because f is coercive on V, $\{u_n\}$ is uniformly bounded on V. Hence $\{u_n\}$ contains a weakly convergent subsequence, which we shall again denote by u_n: $u_n \rightharpoonup \hat{u}$ in V. As t is weakly continuous we have $t(\hat{u}) = \lim t(u_n) = p$. Moreover, as f is weakly lower semicontinuous, we have

$$f(\hat{u}) \leq \lim\inf f(u_n) = \lim h(q_n) = h(p).$$

Hence, $f(\hat{u}) \leq h(p)$ and $t(\hat{u}) = p$ so that by definition of h we must have $f(\hat{u}) = h(p)$ and $t(\hat{u}) = p$, which shows that $\hat{u} \in P_p$. From this statement (3.6) follows. With condition (f3) it follows from $f(\hat{u}) = \lim f(u_n)$ and $u_n \rightharpoonup \hat{u}$, $u_n \to \hat{u}$, which proves (3.7).

Corollary 3.5. Suppose f satisfies the extra condition (f3). If

$$\{u_n\} \subset \bigcup_{q\in[\alpha,\beta]} P_q$$

is a convergent sequence, $u_n \to \hat{u}$ say, with $t(u_n) = q_n \to p \in [\alpha,\beta]$ and $q_n \in (\alpha,\beta)\forall_n$, then $\hat{u} \in P_p$ satisfies

$$f'(\hat{u}) = \hat{\mu} t'(\hat{u}),$$

wherein

$$\hat{\mu} = \lim \mu_n := \lim \frac{\mathrm{d}h}{\mathrm{d}p}(q_n)$$

[thus $\hat{\mu} = (\mathrm{d}h/\mathrm{d}p)(p)$ if $p \in (\alpha,\beta)$ and $\hat{\mu} = h'_+(\alpha)$ if $p = \alpha$, $\hat{\mu} = h'_-(\beta)$ if $p = \beta$].

Proof. The elements u_n satisfy $f'(u_n) = \mu_n t'(u_n)$ where $\mu_n = (\mathrm{d}h/\mathrm{d}p)(q_n)$ because of Corollary 3.2. Put $\hat{\mu} = \lim \mu_n$, then the results follow by continuity.

Remark 3.6. Note that if f is a quadratic functional (as will be case in the applications of Sec. 5), condition (f3) is nothing but the compactness condition c.c. of Lemma 1.2. In a more general setting, it can be seen from the proofs of Theorem 3.4 and Corollary 3.5, that condition (f3) may be replaced by the following Palais–Smale-type condition:

(f3)* for every sequence $\{u_n\} \subset V$ for which $f'(u_n) - \mu t'(u_n) \to 0$ in V^* and for which $u_n \rightharpoonup \hat{u}$ in V, it follows that $u_n \to \hat{u}$ in V.

Indeed, for the sequence $\{u_n\}$ considered in the foregoing proofs we have $u_n \rightharpoonup \hat{u}$ and $f'(u_n) = \mu_n t'(u_n)$ with $\mu_n = (\mathrm{d}h/\mathrm{d}p)(q_n)$. As t is weakly continuous we have $t'(u_n) \rightharpoonup t'(\hat{u})$ (and in fact also strong convergence) in V^*, and thus

$$f'(u_n) - \hat{\mu} t'(u_n) = (\mu_n - \hat{\mu}) t'(u_n) \to 0 \text{ in } V^*$$

where $\hat{\mu} = \lim \mu_n$. With (f3)* it then follows that $u_n \to \hat{u}$ in V, and consequently $f'(u_n) - \hat{\mu} t'(u_n) \to f'(\hat{u}) - \hat{\mu} t'(\hat{u}) = 0$.

If h is continuous at p_0 but not convex in a neighbourhood we must face the possibility that h has a corner there. From Corollary 3.5 it follows that [if f satisfies ($f3$)] in this case both $h'_+(p_0)$ and $h'_-(p_0)$ are multipliers of solution of $\mathcal{P}_{p_0}$. Hence, in this case, $\mathcal{P}_{p_0}$ has at least two different solutions with multipliers $h'_+(p_0)$ and $h'_-(p_0)$, solutions which are limits of solutions "from above" and "from below," respectively. From this observation we arrive at the converse of Corollary 3.2:

Corollary 3.7. Suppose f satisfies ($f3$) and let h be continuous at p_0. Then if the solution of $\mathcal{P}_p$ is unique, or if *all* solutions of $\mathcal{P}_{p_0}$ have the same multiplier, μ_0, then h is differentiable at p_0 with $\mu_0 = (\mathrm{d}h/\mathrm{d}p)(p_0)$.

4. BRANCH OF CONSTRAINED EXTREMAL SOLUTIONS

Let us introduce the notion of nondegeneracy for (solutions of) problem $\mathcal{P}_p$ in the following way:

Definition 4.1. A solution u_0 of $\mathcal{P}_p$ is called *nondegenerate* if u_0 satisfies conditions (i) and (ii) of Theorem 2.1 wherein μ_0 is the unique multiplier corresponding to u_0. For $p \in t(V)$, problem $\mathcal{P}_p$ is said to be nondegenerate if every solution of $\mathcal{P}_p$ is nondegenerate.

If we call any solution of $f'(u) = \mu t'(u)$ for which $t'(u) \neq 0$ a constrained stationary point of f, the contents of Theorem 2.1 may be stated as follows: if u_0 is a nondegenerate solution of $\mathcal{P}_{p_0}$, then u_0 lies on a unique, smooth curve of constrained stationary points of f. As has already been remarked, this curve needs not to consist of constrained extremal elements, i.e., of solutions of $\mathcal{P}_p$. We shall now describe a simple situation for which all elements of this curve are in fact solutions of $\mathcal{P}_p$. But first a result about the number of solutions of $\mathcal{P}_p$.

Lemma 4.1. If u_0 is a nondegenerate solution of $\mathcal{P}_p$, then u_0 is an isolated constrained stationary point on the level set $t^{-1}(p)$.

Proof. Immediate from the uniqueness of the continuation through (u_0,μ_0): if $\{u_n\}$ is a sequence of constrained stationary points of f on $t^{-1}(p_0)$ which converges to u_0, then for sufficiently large n, $t'(u_n) \neq 0$ and then there are unique multipliers μ_n corresponding to u_n such that $f'(u_n) = \mu_n t'(u_n)$. As $f'(u_n) \to f'(u_0)$ and $t'(u_n) \to t'(u_0)$ for $n \to \infty$, $\mu_n \to \mu_0$ for $n \to \infty$, conflicting the uniqueness statement in Theorem 2.1.

THEOREM 4.2. *Suppose f satisfies ($f3$) and assume that problem $\mathcal{P}_p$ is nondegenerate. Then the number of solutions of $\mathcal{P}_p$ is finite.*

Proof. If f satisfies ($f3$), P_p is a compact set (see Ref. 5, Proposition 2.2) and, according to the foregoing lemma, consists of isolated elements.

THEOREM 4.3. *Let f satisfy ($f3$) and suppose that an interval $J \subset t(V)$ can be found for which h is continuous on J and such that for every $p \in J$, $\mathcal{P}_p$ has precisely* one *nondegenerate solution, say $U(p)$ with multiplier, $\mu(p) = (\mathrm{d}h/\mathrm{d}p)(p)$.*
Then $J \ni p \to U(p)$ is a continuously differentiable mapping in V.

Proof. Let $p_0 \in J$ and consider $(u_0,\mu_0) \in V \times \mathbf{R}$, where $u_0 = U(p_0)$ and $\mu_0 = \mu(p_0)$. According to Theorem 2.1 there exists a smooth continuation of constrained stationary points, say $(\dot{u}(p),\dot{\mu}(p))$ with $\dot{u}(p) \to u_0$ and $\dot{\mu}(p) \to \mu_0$ if $p \to p_0$, which continuation is unique. As h is differentiable (Corollary 3.7), $\dot{\mu}(p) = \mu(p)$. We shall show that for every p with $|p - p_0|$ sufficiently small, $\dot{u}(p)$ is in fact the unique solution of $\mathcal{P}_p : \dot{u}(p) = U(p)$. To that end suppose the contrary: suppose there is a sequence $q_n \to p_0$ with $\dot{u}(q_n) \notin P_{q_n}$. Consider the sequence $\{U(q_n),\mu(q_n)\}_{n\in\mathbf{N}}$. Because of Theorem 3.4, for some subsequence $q_{n'}$, we have $U(q_{n'}) \to u_0$ and $\mu(q_{n'}) \to \mu_0$ which conflicts the uniqueness of the continuation through (u_0,μ_0). Hence $\dot{u}(q_n) = U(q_n)$. As the continuation described in Theorem 2.1 is continuously differentiable, it follows that $U(p)$ is continuously differentiable at p_0, for every $p_0 \in J$, which proves the theorem.

Note that if in the foregoing theorem we can take for J the whole range of t, we have obtained a global continuation of constrained extremal solutions, smoothly described by the parameter p.

Remark 4.4. When dealing with symmetric functionals f and t, then, if $u \in P_p$, $-u \in P_p$. Hence for such functionals the conditions of Theorem 4.3 are not satisfied for any $J \subset t(V)$. However in a specific situation the foregoing results may be easily modified. Therefore suppose that for $p \in J$, $P_p = \{u(p), -u(p)\}$, and that there exists a subset $K \subset V$ such that $u(p) \in K$ but $-u(p) \notin K\ \forall p \in J$ (e.g., K may be the cone of positive functions in a specific example). Then, for $p \in J$, $u(p)$ is the unique solution of the modified problem

$$\mathscr{P}_p(K): \inf_{u \in t^{-1}(p) \cap K} f(u)$$

[the restriction $u \in K$ in $\mathscr{P}_p(K)$ is a "natural constraint" for the specific solution $u(p)$of $\mathscr{P}_p$]. Then the foregoing results are valid for problems $\mathscr{P}_p(K)$.

5. APPLICATION

To demonstrate the foregoing results we shall investigate in this section a class of specific problems of semilinear elliptic-type. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and L a uniformly elliptic operator of the form

$$L = -\sum_{1 \le i, j \le n} \partial_{x_i}[a_{ij}(x)\partial_{x_j}] + c(x),$$

where the coefficients of L are real, $a_{ij}(x) = a_{ji}(x)$ is twice continuously differentiable in $\overline{\Omega}$ for $1 \le i, j \le n$ and $c(x)$ is nonnegative and once continuously differentiable in $\overline{\Omega}$. With $V = \mathring{H}^1(\Omega) = \mathring{W}^{1,2}(\Omega)$ the usual Sobolev space, the functional

$$f(u) := \tfrac{1}{2}\langle u, Lu\rangle = \tfrac{1}{2}\int_\Omega u(x) \cdot Lu(x)\mathrm{d}x \tag{5.1}$$

leads to the (nonlinear) eigenvalue problem

$$\begin{aligned} Lu &= \mu t'(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

It is a standard result that f defined by (5.1) satisfies condition (f1,2). Moreover, f is equivalent with the square of the norm in $H^1(\Omega)$: for some $\alpha > 0$

$$\frac{1}{\alpha}\ \| u \|^2_{H^1} \le f(u) \le \alpha\ \| u \|^2_{H^1},$$

and satisfies (f3).

Consider a functional t of the form

$$t(u) = \int_\Omega \Gamma(x, u(x))\mathrm{d}x, \text{ with } \Gamma(x,z) := \int_0^z \gamma(x,t)\mathrm{d}t, \tag{5.2}$$

where $\gamma \in C^3(\Omega \times \mathbf{R}, \mathbf{R})$ is a given function. From standard embedding results for $\mathring{H}^1(\Omega)$

it follows that the functional t is defined on V and satisfies condition $(t1,2)$ if γ satisfies the following growth conditions:

($t3$) If $n > 2$, then $|\gamma(x,z)| \le b_1 + b_2|z|^s$ for $z \in \mathbf{R}$ where $s < (n+2)/(n-2)$, and b_1, b_2 are positive constants.
If $n = 2$ then $\gamma(x,z) \le \exp\chi(z)$, where

$$\lim_{z\to\infty} \frac{\chi(z)}{z^2} = 0.$$

We will assume this condition to be satisfied in the following, and in this case for every $p \in t(V)$, $\mathcal{P}_p$ has at least one solution u, and solutions for which $\gamma(x,u) \not\equiv 0$ in Ω satisfy for some $\mu \in \mathbf{R}$ the equation

$$\begin{aligned} Lu &= \mu\gamma(x,u) \quad \text{in } \Omega \\ u &= 0 \qquad\qquad \text{on } \partial\Omega \end{aligned} \tag{5.3}$$

Several examples of this kind were considered in another paper [5]. Here we shall only consider functions γ which satisfy the conditions ($\gamma1$)–($\gamma3$) or ($\gamma1$)–($\gamma5$):

$$(\gamma 1)\colon\quad \gamma(x,0) > 0$$

$$(\gamma 2)\colon\quad \gamma_z(x,0) > 0$$

$$(\gamma 3)\colon\quad \gamma_{zz}(x,z) > 0 \quad \text{for } z > 0$$

$$(\gamma 4)\colon\quad \lim_{z\to\infty} \frac{\gamma(z)}{z} = \infty$$

$$(\gamma 5)\colon\quad \Gamma(x,z) = \int_0^z \gamma(x,t)\,dt \le \theta z\gamma(x,z) \quad \text{for } z > \bar{z},$$

for some $\bar{z} > 0$ and $\theta \in [0,\frac{1}{2})$.

With (some of) these assumptions, positive solutions of (5.3) have been extensively studied (see, e.g., Crandall and Rabinowitz [2], Keener and Keller [6], and the references therein). The main results are summarized in the following:

THEOREM 5.1. *Suppose γ satisfies (γ1)–(γ3). Then*

(*i*) *if (u,μ) is a solution of (5.3) with $\mu > 0$ and $u \ge 0$, then $\mu < \mu_1$, where μ_1 is the least eigenvalue of*

$$\begin{aligned} Lv &= \mu\gamma_z(x,0)v \text{ in } \Omega \\ v &= 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned}$$

(*ii*) *There exists a (maximal) number $\bar{\lambda} \in (0,\mu_1]$ and a smooth curve $\{U(\mu) \mid 0 \le \mu < \bar{\lambda}\} \subset C^{2,\alpha}$, with $0 < \alpha < 1$, of minimal positive solutions [i.e., if $u(\mu)$ is another nonnegative solution of (5.3), then $U(\mu)(x) \le u(\mu)(x)$ for $x \in \Omega$). Moreover, the least eigenvalue of the operator $Q(U(\mu),\mu)$ defined by (1.3) is positive:*

$$\sigma_1(Q(U(\mu),\mu)) > 0, \qquad 0 < \mu < \bar{\lambda}$$

whereas

$$\sigma_1(Q(u(\mu),\mu)) < 0, \qquad 0 \le \mu \le \bar{\lambda}$$

for every other solution of (5.3).

(*iii*) *If* $\| U(\mu) \|_{C(\bar{\Omega})} \le M$ *for some constant* M *for each* $\mu \in (0, \bar{\lambda})$, *then*

$$\bar{U} := \lim_{\mu \uparrow \bar{\lambda}} U(\mu)$$

exists (in $C^{2,\alpha}$*), and* $\sigma_1(Q(\bar{U}, \bar{\lambda})) = 0$ *with a nonnegative (unique) eigenfunction:* $Q(\bar{U}, \bar{\lambda})v(x) = 0$, $v(x) \ge 0$ *in* Ω. *In this case there is a "bending" of the solution curve at* $(\bar{U}, \bar{\lambda})$.

(*iv*) *If* γ *satisfies also* (γ4,5), *then for every* $\mu \in (0, \bar{\lambda})$ *there are at least two nonnegative solutions of (5.3).*

For the proof of these results the reader is referred to Crandall and Rabinowitz [2]. In order to interpret these results in terms of the foregoing theory, we have to assume an extra condition which assures that the solutions of problem $\mathcal{P}_p$, $p > 0$ are nonnegative. A simple condition which fulfills this requirement will be seen to be

$$(\gamma 0) \qquad \gamma(x,z) > 0 \qquad \forall z \in \mathbf{R},$$

which implies that $t(|u|) \ge t(u) \quad \forall u \in V$.

The results of the next lemma were proved in another paper [5].

Lemma 5.2. Assume γ satisfies (γ0)–(γ3). Then $[0,\infty) \subset t(V)$ and, for every $p > 0$, problem $\mathcal{P}_p$ in equivalent with the "inverse" constrained extremum problem $\mathcal{S}_{h(p)}$, where, for $r > 0, \mathcal{S}_r$ is the problem

$$\mathcal{S}_r\text{:} \ \sup_{u \in f^{-1}(r)} t(u) = \sup_{u \in \{f^{-1}(\rho) | 0 < \rho \le r\}} t(u)$$

(i.e., u is a solution of $\mathcal{P}_p$ if and only if u is a solution of $\mathcal{S}_{h(p)}$), and the function

$$s(r) := \sup_{u \in f^{-1}(r)} t(u)$$

is the inverse of the function $h(p)$:

$$h(s(r)) = r \text{ and } s(h(p)) = p \text{ for } p > 0, r > 0.$$

Furthermore, $h(0) = 0$ and $h : [0,\infty) \to \mathbf{R}$ is continuous and monotonically increasing, and hence the multiplier $\mu(p)$ of every solution of $\mathcal{P}_p$ is positive. Because of (γ0), every solution of ($\mathcal{S}_r$ and hence of) $\mathcal{P}_p$, $p > 0$, is nonnegative on Ω (and hence from the maximum principle, positive in Ω).

Because of condition (γ1) the multiplier of a solution of $\mathcal{P}_p$ tends to zero if $p \downarrow 0$. As $h(0) = 0$, $h(p) > 0$ for $p > 0$ and h is continuous, there exists some maximal value p^* such that h is convex, and hence differentiable, on $(0,p^*)$. It may happen that $p^* = \infty$. In that case problem $\mathcal{P}_p$ is equivalent with the unconstrained minimum problem

$$\inf_{u \in V} \{(f(u) - \mu(p)t(u)\} \tag{5.4}$$

where $\mu(p) = (\mathrm{d}h/\mathrm{d}p)(p)$ (see Ref. 5, Corollary 4.9). Then, because of Theorem 5.1 (i), (ii), $U(\mu(p))$ is the unique solution of (5.4) and $\mu(p) \uparrow \bar{\lambda}$ monotonically as $p \to \infty$, and hence $h(p) \uparrow \bar{\lambda} p$ + const. as $p \to \infty$. Note that in this case the boundedness condition of Theorem 5.1 (iii) is certainly not satisfied.

If γ satisfies (γ4), $p^* < \infty$. This follows from the observation that in this case for an arbitrary smooth, positive function w on Ω, $f(\rho w) - \mu t(\rho w) \to -\infty$ if $\rho \to \infty$, for every $\mu > 0$. Hence

$$\inf_{u \in V} f(u) - \mu t(u) = -\infty$$

for every $\mu > 0$. Consequently [5] $h(p)$ is not subdifferentiable for $p > 0$ and hence h is not convex on $[0,\infty)$. In fact, if γ satisfies (γ4,5), we can summarize our knowledge of the problem as follows

THEOREM 5.2. *Let γ satisfy (γ0)–(γ5). Then we have*

(*i*) *For every $p > 0$: $\mathcal{P}_p$ has at least one solution, every solution is nonnegative; the function $h : [0,\infty) \to [0,\infty)$ is continuous and monotonically increasing.*

(*ii*) *There exists a finite value $p^* > 0$ such that*

(*a*) *for $0 < p \leq p^*$: $h(p)$ is convex, continuously differentiable, and $\mu(p)$ increases from 0 to $\bar{\lambda} = \mu(p^*)$; the solution of $\mathcal{P}_p$ is unique and is the minimal positive solution $U(\mu(p))$ [Hence Theorem 4.1 applies with $J = (0,p^*)$].*

(*b*) *For $p > p^*$, $h(p)$ is a concave function, and μ monotonically decreases from $\bar{\lambda}$ to 0 if $p \to \infty$. For every solution $u(p)$ of $\mathcal{P}_p$ the least eigenvalue of $Q(u(p),\mu(p))$ is negative.*

(*c*) *At $p = p^*$ we have $\mu(p^*) = \bar{\lambda}$ and $(\mathrm{d}\mu/\mathrm{d}p)(p^*) = 0$. Furthermore, there exists a continuation of constrained stationary points of f through the bending point $(\bar{U}, \bar{\lambda})$, which continuation is smoothly described with the parameter p, and for $0 < p < p^*$ this continuation is given by $U(\mu(p),\mu(p))$.*

Proof. Apart from what has already been shown, the proof is as follows:

(ii)(a). From Ref. 5, Proposition 4.8 it follows that as h is convex, solutions of $\mathcal{P}_p$, $0 < \mu < p^*$ are local minimal points of the functional $f - \mu(p)t$. Hence for these solutions we have $\sigma_1(Q(u(p),\mu(p)) \geq 0$, and then by Theorem 5.1(ii) $u(p) = U(\mu(p))$ is a solution of $\mathcal{P}_p$, which solution is, moreover, unique. With Theorem 4.3 we obtain a continuously differentiable curve $\{ U(\mu(p),\mu(p)) \mid 0 < p < p^*\} \subset V \times \mathbf{R}$ which, by continuity, coincides with the continuously differentiable curve $\{ U(\mu),\mu) \mid 0 < \mu < \bar{\lambda}\}$. Hence $\mu(p^*) = \bar{\lambda}$ and (ii)(a) is proved.

(ii)(b). Any solution $u(p)$ of $\mathcal{P}_p$ with $p > p^*$ must have $\sigma_1(Q(u(p),\mu(p))) < 0$ because of Theorem 5.1(ii). It then follows from Corollary 4.14 of Ref. 5 that h is concave for $p > p^*$. Hence $\mu(p)$ monotonically decreases, and because of Theorem 5.1(iv) $\mu(p) \to 0$ if $p \to \infty$.

(ii)(c). As $h(p^*)$ is finite, it follows from Theorem 5.1(iii) that $\bar{U}$ exists, and that $\bar{U}$ is a nondegenerate solution of $\mathcal{P}_{p^*}$: as the (simple) eigenfunction $v(x)$ is nonnegative,

$$\int_\Omega \gamma(x, \bar{U}(x)) \cdot v(x)\, \mathrm{d}x > 0,$$

which shows that $v \notin \tau_{\bar{U}}$ and hence $\nu(p^*) > 0$ where ν is the number defined by (1.6). Hence according to Theorem 2.1 there exists a smooth continuation through $(\bar{U}, \bar{\lambda})$ which continuation is for $0 < p < p^*$ given in (ii)(a). Hence μ is differentiable at p^*, and as

$$\lim_{p \uparrow p^*} \frac{d\mu}{dp^*}(p) = 0,$$

we have $(d\mu/dp)(p^*) = 0$.

Remark 5.4. Note that in Theorem 5.3 we encounter the bending of the solution curve as described in Remark 2.3.

Remark 5.5. To conclude we mention that in special situations it is known that (5.3) has precisely two nonnegative solutions for every $\mu \in (0,\bar{\lambda})$ {e.g., if $n = 2$ and Ω radially symmetric (or if $n = 1$) with $L = -\Delta$ and $\gamma(z) = e^z$, see Gelfand [4]}. In such a case, $\mathscr{P}_p$ has for every $p > 0$ a unique solution and Theorem 4.3 applies with $J = [0,\infty)$: the parameter p is a global parameter of which all the constrained extremal solutions depend in a continuously differentiable way.

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